Violations of local realism by two entangled quNits

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Results obtained in two recent papers, [1] and [2] seem to indicate that the nonlocal character of the correlations between the outcomes of measurements performed on entangled systems separated in space is not robust in the presence of noise. This is surprising, since *entanglement* itself is robust. Here we revisit this problem and argue that the class of gedanken-experiments considered in [1] and [2] is too restrictive. By considering a more general class, involving *sequences* of measurements, we prove that the nonlocal correlations are in fact robust.

In his famous paper [3], J. Bell showed that quantum mechanics predicts nonlocal correlations between measurement outcomes at spatially separated regions in a certain experiment. By nonlocal correlations we mean correlations which cannot be explained by any local hidden variable model (LHV). During the last few years other aspects of nonlocality, in addition to generating nonlocal correlations have been discovered. For example, the ability of quantum states to teleport [4], to superdense code [5], and to reduce the number of classical bits required to perform certain communication tasks (in the so called “communication complexity” scenario) [6]. Further, nonlocality appears to be at the heart of quantum computation [7] and its ability to perform certain computations exponentially faster than any classical device.

Two recent papers [1] and [2] have studied the question of robustness of nonlocal correlations. Results in [1] and [2] seem to indicate a very surprising result. Namely, it appears that in a certain sense (which we will define more precisely later), quantum nonlocal correlations are not very robust. Here we would like to argue that nonlocal correlations are actually very robust. While we do not disagree with the specific results found in [1] and [2], we show that the class of gedanken experiments they have considered (though very interesting in itself) is in fact quite limited and not sensitive enough. We present a different class of experiments which shows that nonlocal correlations are robust.

The authors of [1] and [2] have considered two quantum particles, each living in an N dimensional Hilbert space, which are in the maximally entangled state mixed with random noise. ie. states of the form

$$\rho_N(F_N) = (1 - F_N) \left|\Psi_N\right\rangle_{AB} \left\langle\Psi_N\right| + F_N \frac{1}{N^2} \hat{I}_{N\times N}, \quad (1)$$

where

$$\left|\Psi_N\right\rangle_{AB} = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} \left|m\right\rangle_A \left|m\right\rangle_B, \quad (2)$$

F_N is a constant $0 \leq F_N \leq 1$ which describes the fraction of noise and $\hat{I}_{N\times N}$ is the identity matrix. They have asked, “what is the maximum fraction of noise, F_N, which can be added to the maximally entangled state so that the state still generates nonlocal correlations?”

It is useful here to make a clear distinction between two different issues which are relevant for our discussion. The first is the issue of *entanglement* or *non-separability.* A quantum state is *separable* if it can be written as

$$\rho_{AB} = \sum_i p_i \rho_A^i \rho_B^i, \quad (3)$$

and it is non-separable otherwise.

It has been shown [8], [9] and [10] that if too much noise is added to the maximally entangled state, the state ceases to be entangled. Obviously, at this moment the quantum state ceases to have any nonlocal aspects whatsoever.

The other issue is whether or not the results of all possible measurements performed on the state can be explained by a local hidden variable model. If they cannot we say, following Bell, that the state generates nonlocal correlations (sometimes this is called a “violation of local realism”).

It is clear that when there is so much noise that the state becomes separable, the state cannot generate any nonlocal correlations. It is however possible that the state ceases to generate nonlocal correlations at smaller levels of noise, i.e. while it is still entangled. Indeed, it is not known if every entangled (mixed) state generates nonlocal correlations or not - this is one of the most important issues in quantum nonlocality.

It appears from the results of [1] and [2] that the nonlocal correlations are not robust, meaning that for fractions of noise greater than $F_N \approx 0.33$ none of the states $\rho(F_N)$ produce nonlocal correlations. This is very surprising since the entanglement property of the maximally entangled states is robust - for any fraction of noise, when the dimensionality of the systems is large enough (how large depending on the fraction of noise), the states of form (1) are entangled. Furthermore, these mixed entangled states exhibit most other nonlocality aspects - for example they can be used for teleportation, super-dense coding, and can be purified to yield singlets. So it would be quite strange if they couldn't also generate nonlocal correlations.

We shall show that nonlocal correlations are, similar to entanglement, robust. More precisely we shall show that for any fraction of noise there are states (and experiments to perform upon those states) which exhibit nonlocal correlations. The reason that [1] and [2] did not find these experiments is because they only looked at experiments in which a single von-Neumann measurement is made on each particle; here we look at *sequences* of von-Neumann measurements.

The present discussion is, to some extent, a repeat of the history concerning Werner's density matrices. In 1989 Werner [11] presented some density matrices which are entangled but which are such that if single von-Neumann measurements are made on each particle, the results can be explained by a local hidden variables model. At that time it was tacitly assumed that performing single von-Neumann measurements on each particle essentially covers all possibilities. However it was subsequently shown [12] that the outcomes of *sequences* of von-Neumann measurements are nonlocal - they cannot be explained by any hidden variables model. This work was then extended in [13], [14] and [15].

We shall next explain why performing sequences of measurements puts additional constraints on local hidden variable models, then use this to prove that there are states with arbitrarily high fractions of noise which exhibit nonlocal correlations.

Consider two observers, Alice and Bob, situated in two space separated regions. The standard assumption of LHV is that if Alice performs any arbitrary measurement A and Bob performs any arbitrary measurement B, and the measurements are timed so that they take place outside the light-cone of each other, then there exists a shared random variable λ, with distribution $\mu(\lambda)$, and local distributions $P_A(a;\lambda)$ and $P_B(b;\lambda)$ such that the joint probability that the measurement of A yields a and the measurement of B yields b is given by

$$P_{AB}(a,b)=\int P_A(a;\lambda)P_B(b;\lambda)\mu(\lambda)d\lambda. \tag{4}$$

Consider now that Alice and Bob, instead of subjecting their particles to a single measurement, perform two measurements one after the other, say A_1 followed by A_2 and B_1 followed by B_2. Then a LHV model implies that

$$\begin{aligned}&P_{A_1A_2B_1B_2}(a_1,a_2,b_1,b_2)=\\&\int P_{A_1A_2}(a_1,a_2;\lambda)P_{B_1B_2}(b_1,b_2;\lambda)\mu(\lambda)d\lambda.\end{aligned} \tag{5}$$

Quantum mechanically the two measurements on each side could be viewed as a single POVM. For LHV models however, doing one measurement after the other gives us the extra constraint that we must be able to write $P_{A_1A_2}(a_1,a_2;\lambda)$ in the form

$$P_{A_1A_2}(a_1,a_2;\lambda)=P_{A_1}(a_1;\lambda)P_{A_2}(a_2;A_1,a_1,\lambda). \tag{6}$$

Here $P_{A_1}(a_1;\lambda)$ is the probability that Alice's particle yields the answer a_1 when the first measurement to which is subjected is A_1 and given that the hidden variable has the value λ. $P_{A_2}(a_2;A_1,a_1,\lambda)$ is the probability that Alice's particle yields the outcome a_2 when the second measurement is A_2, given that the hidden variable has the value λ and given that it was first subjected to a measurement of A_1 to which it yielded the outcome a_1. The reason is that when Alice's particle has to give the outcome of measurement A_1, it does not yet know what exactly will be the measurement A_2 that will be subsequently performed, and so cannot use that information to decide which outcome a_1 to give. We must write Bob's probabilities in a similar way.

Now, let us look at the probabilities of outcomes of the second measurement, conditioned on some fixed result of the first.

$$P_{A_2B_2}(a_2,b_2;A_1,a_1,B_1,b_1)=\frac{P_{A_1A_2B_1B_2}(a_1,a_2,b_1,b_2)}{P_{A_1B_1}(a_1b_1)}. \tag{7}$$

Substituting (5) and (6) into (7), and defining

$$\tilde{\mu}(\lambda)=\frac{P_{A_1}(a_1;\lambda)P_{B_1}(b_1;\lambda)}{\int P_{A_1}(a_1;\lambda)P_{B_1}(b_1;\lambda)\mu(\lambda)d\lambda}, \tag{8}$$

we have that

$$\begin{aligned}&P_{A_2B_2}(a_2,b_2;A_1,a_1,B_1,b_1)=\\&\int P_{A_2}(a_2;A_1,a_1,\lambda)P_{B_2}(b_2;B_1,b_1,\lambda)\tilde{\mu}(\lambda)d\lambda.\end{aligned} \tag{9}$$

We shall now only consider experiments in which the first measurements are fixed and give some particular fixed outcomes, and thus can drop the indices A_1, a_1, B_1 and b_1, which leaves us with

$$P_{A_2B_2}(a_2b_2)=\int P_{A_2}(a_2;\lambda)P_{B_2}(b_2;\lambda)\tilde{\mu}(\lambda)d\lambda. \tag{10}$$

We further note that $\tilde{\mu}(\lambda)$ is positive and $\int\tilde{\mu}(\lambda)d\lambda=1$, thus it can be viewed as a probability distribution analogously to $\mu(\lambda)$. Thus, if the whole experiment could be explained by a local hidden variables model, then the probabilities of outcomes for the second measurement conditioned upon any result of the first measurement have to be given by a LHV model themselves. This is a consequence of doing the measurements one after the other rather than together. In particular, we can look at

Bell inequalities for these conditioned probabilities, and know that if they are violated, then the initial state is nonlocal. For example suppose that the second measurement which is performed by Alice is either A_2 or A'_2 and that performed by Bob is either B_2 or B'_2. Then using the CHSH inequality [16] (a particular Bell type inequality) and (10) it follows that

$$E(A_2B_2) + E(A_2B'_2) + E(A'_2B_2) - E(A'_2B'_2) \leq 2. \quad (11)$$

Here $E(A_2B_2) = Tr\tilde{\rho}A_2B_2$ is the expectation value of the product of the operators A_2 and B_2 in the state $\tilde{\rho}$ which is the state of the system after the first measurements (assuming that we indeed obtained the particular fixed outcomes we have chosen).

We shall now use (11) to show that for sufficiently large N, the states defined in equation (1) generate nonlocal correlations. We take the first measurement on Alice's side, A_1, to be the projection onto the subspace $\{|1>_A, |2>_A\}$. The first measurement on Bob's side, B_1, is the projection onto the subspace $\{|1>_B, |2>_B\}$. We just look at the cases where the state is indeed in the first two subspaces, in which case the state becomes (after the first measurements):

$$\tilde{\rho} = \frac{(1-F_N)N}{N(1-F_N)+2F_N} \left|\Psi_2\right\rangle \left\langle\Psi_2\right| + \frac{2F_N}{N(1-F_N)+2F_N} \frac{\hat{I}_{2\times 2}}{2^2}. \quad (12)$$

We now take the second measurements (A_2, A'_2, B_2, B'_2) to be those which give the maximal violation of the CHSH inequality on the state $|\Psi_2\rangle_{AB}$, and we note that if the CHSH inequality is violated, the initial state is nonlocal. This occurs when

$$F_N < \frac{N}{N+c}, \quad (13)$$

where $c = \frac{2}{\sqrt{2}-1} \approx 4.83$. Therefore, for any fraction of noise we can, by taking N large enough, find states which give nonlocal correlations. Thus we have shown that the nonlocal correlations are robust to noise.

Finally, we note that we have not completely solved the problem of which states of the form (1) generate nonlocal correlations. Recalling that [7-9] states of this form are separable iff $F_N \geq \frac{N}{N+1}$, we can see that the states for which $\frac{N}{N+c} \leq F_N < \frac{N}{N+1}$ are entangled but do not violate the Bell inequality we have considered. It is an interesting and open question as to whether these states generate nonlocal correlations or not.

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